

The van Est spectral sequence for Hopf algebras

E. J. Beggs & Tomasz Brzeziński

Department of Mathematics
University of Wales, Swansea
Wales SA2 8PP

Abstract. Various aspects of the de Rham cohomology of Hopf algebras are discussed. In particular, it is shown that the de Rham cohomology of an algebra with the differentiable coaction of a cosemisimple Hopf algebra with trivial 0-th cohomology group, reduces to the de Rham cohomology of (co)invariant forms. Spectral sequences are discussed and the van Est spectral sequence for Hopf algebras is introduced. A definition of Hopf-Lie algebra cohomology is also given.

1 Introduction

The idea of calculating the de-Rham cohomology of a manifold directly from the definition is a rather daunting one. Fortunately there are all sorts of tools, from the Meyer Vietoris exact sequence to the equivalence of many types of cohomology given by sheaf theory, to assist. The situation in noncommutative geometry is not so fortunate in this regard. In this paper we will give some results on the de Rham cohomology of Hopf algebras equipped with bicovariant differential calculi [22]. These results will be the noncommutative geometry analogues of well known results about Lie groups.

The paper [11] considered the connection between the de Rham cohomology of a Lie group and the cohomology of the left invariant differential forms. In the simplest case, the cohomologies coincide for a compact connected Lie group, with compactness appearing in the guise of a normalised Haar integral. The cohomology of the left invariant differential forms can be taken to define the Lie algebra cohomology [15]. The van Est spectral sequence [6] is a more general result, whose statement requires the additional concept of group cohomology [7]. The involvement of the Lie algebra cohomology is fortunate in that for a (usual finite dimensional) Lie algebra finding it only requires a finite dimensional computation.

In this paper we give results for Hopf algebras (with bicovariant differential calculus) corresponding to the results for Lie groups. First we show that if a Hopf algebra has a normalised left integral, then its de Rham cohomology is isomorphic to the cohomology of left invariant forms. Then we prove a version of the van Est spectral sequence. It then remains to identify the cohomology of left invariant forms in an easily calculable form. This must be, by definition, the cohomology of the Hopf-Lie algebra of the vector fields on the Hopf algebra with given bicovariant differential structure. For Hopf-Lie or braided Lie algebras see [22, 18]. The usual definition of Lie algebra cohomology must be modified, as the Lie bracket no longer satisfies the Jacobi identities.

Throughout the paper we work with vector spaces, algebras etc. over a commutative field k . The unadorned tensor product is always over k . Given an algebra A , $\Omega A = \bigoplus_{n \geq 0} \Omega^n A$ denotes a graded differential algebra with the differential $d : \Omega^n A \rightarrow \Omega^{n+1} A$ such that $A = \Omega^0 A$. Each of the Ω^A is an A -bimodule and we always assume that ΩA satisfies the *density condition*, i.e., that for all $\omega \in \Omega^1 A$ there exist $a_i, b_i \in A$, $i = 1, 2, \dots, m$ such that $\omega = \sum_{i=1}^m a_i db_i$. We often refer to ΩA as a *differential calculus* or a *differential structure on A* . For any Hopf algebra P , the coproduct is denoted by Δ , the counit by ϵ and the antipode by S . We use the Sweedler sigma-notation for a coproduct (without a sigma), i.e., we write $\Delta(p) = p_{(1)} \otimes p_{(2)}$ (summation understood). A left P -coaction on F , $\lambda : F \rightarrow P \otimes F$ is denoted on elements by the Sweedler notation as well, but with indices in square brackets, i.e., $\lambda(f) = f_{[-1]} \otimes f_{[0]}$ (summation understood). The coassociativity of λ entails that for all $f \in F$, $f_{[-1](1)} \otimes f_{[-1](2)} \otimes f_{[0]} = f_{[-1]} \otimes f_{[0](-1)} \otimes f_{[0][0]}$, thus we simply write $f_{[-2]} \otimes f_{[-1]} \otimes f_{[0]}$ etc. For the theory of bicovariant differential calculi on a Hopf algebra we refer to [22] or to [16].

2 Coactions on the de Rham cohomology

In this section we suppose that P is a Hopf algebra with a bicovariant differential calculus, and that P left coacts on an algebra M (with a given differential structure) by a differentiable map $\lambda : M \rightarrow P \otimes M$. The differentiability just means that λ extends to a map $\lambda_* : \Omega^n M \rightarrow \Omega^n(P \otimes M)$ which commutes with the differential d (i.e. a cochain map).

Remark 2.1 By definition of the tensor product differential structure,

$$\Omega^n(P \otimes M) = (\Omega^0 P \otimes \Omega^n M) \oplus (\Omega^1 P \otimes \Omega^{n-1} M) \oplus \dots \oplus (\Omega^n P \otimes \Omega^0 M),$$

and the differential d on $\Omega^n(P \otimes M)$ corresponds to $d \otimes \text{id} + (-1)^r \text{id} \otimes d$ on $\Omega^r P \otimes \Omega^s M$. We define projections $\Pi_r : \Omega^n(P \otimes M) \rightarrow \Omega^r P \otimes \Omega^{n-r} M$. Then there is a left P -coaction on $\Omega^n M$ given by $\bar{\lambda} = \Pi_0 \circ \lambda_* : \Omega^n M \rightarrow P \otimes \Omega^n M$. By definition of d on the tensor product, $\Pi_0(d\xi) = (\text{id} \otimes d)\Pi_0(\xi)$ for $\xi \in \Omega^*(P \otimes M)$ and from this we see that $d : \Omega^n M \rightarrow \Omega^{n+1} M$ is a left P -comodule map. It follows that there is a left P -coaction $\tilde{\lambda} : H_{dR}^*(M) \rightarrow P \otimes H_{dR}^*(M)$ given by $[\omega] \mapsto (\text{id} \otimes [\bullet])\bar{\lambda}(\omega)$.

The next proposition could be regarded as a part of a noncommutative Künneth theorem.

Proposition 2.2 *The image of the left coaction $\tilde{\lambda} : H_{dR}^n(M) \rightarrow P \otimes H_{dR}^n(M)$ is contained in $(\ker d : P \rightarrow \Omega^1 P) \otimes H_{dR}^n(M)$.*

Proof Given $\omega \in \Omega^n M$ with $d\omega = 0$, we set $\Pi_0(\omega) = \sum_i p_i \otimes \tau_i \in P \otimes \Omega^n M$ and $\Pi_1(\omega) = \sum_j \xi_j \otimes \eta_j \in \Omega^1 P \otimes \Omega^{n-1} M$. Since $d\omega = 0$ we have $0 = \sum_i p_i \otimes d\tau_i \in P \otimes \Omega^{n+1} M$ and $0 = \sum_i dp_i \otimes \tau_i - \sum_j \xi_j \otimes d\eta_j \in \Omega^1 P \otimes \Omega^n M$. Without loss of generality, from the first equality we may assume that all $\tau_i \in \ker d : \Omega^n M \rightarrow \Omega^{n+1} M$. Now using the quotient map $[\bullet] : (\ker d : \Omega^n M \rightarrow \Omega^{n+1} M) \rightarrow H_{dR}^n(M)$, the second equation gives $\sum_i dp_i \otimes [\tau_i] = 0 \in \Omega^1 P \otimes H_{dR}^n(M)$. It follows that $\sum_i p_i \otimes [\tau_i] \in (\ker d : P \rightarrow \Omega^1 P) \otimes H_{dR}^n(M)$. \square

Definition 2.3 Let P be a Hopf algebra with a given bicovariant differential structure. P is called a *connected Hopf algebra* if $H_{dR}^0(P) = k$. The unit element $1 \in k$ is identified with the class of the identity 1_P in $H_{dR}^0(P)$.

Note that the notion of connectedness introduced here is *differential calculus dependent*, i.e., a Hopf algebra can be a connected Hopf algebra with respect to a given differential structure and does not have to be a connected Hopf algebra with respect to another differential structure. For example the quantum group $SU_q(2)$ is a connected Hopf algebra with respect to the 4D-differential calculi of Woronowicz (cf. [22]). The quantum group $GL_q(2)$ is not connected with respect to these calculi, as the quantum determinant induces a non-trivial class in the de Rham cohomology. On the other hand, any Hopf algebra (over a field) is a connected Hopf algebra with respect to the universal differential structure.

Corollary 2.4 *If P is a connected Hopf algebra (with its given bicovariant differential structure), then all elements of $H_{dR}^*(M)$ are fixed by the coaction $\tilde{\lambda}$.*

In [8] it was shown that $H_{dR}^0(P)$ is a Hopf algebra. We have shown that $([\bullet] \otimes \text{id})\tilde{\lambda} : H_{dR}^*(M) \rightarrow H_{dR}^0(P) \otimes H_{dR}^*(M)$ is a left $H_{dR}^0(P)$ -coaction. We can go further to a coaction of the entire graded Hopf algebra $H_{dR}^*(P)$, also described in [8]. To do this we will state the Künneth theorem for noncommutative de Rham cohomology. Its proof is standard and straightforward homological algebra, but it is useful to state it in this context.

Theorem 2.5 *Let M and N be algebras with differential calculi, and give $N \otimes M$ the standard tensor product differential calculus. Then there is an isomorphism*

$$\bigoplus_{n \geq r \geq 0} H_{dR}^r(N) \hat{\otimes} H_{dR}^{n-r}(M) \cong H_{dR}^n(N \otimes M)$$

given by mapping $[\omega] \hat{\otimes} [\xi] \in H_{dR}^r(N) \hat{\otimes} H_{dR}^{n-r}(M)$ to $[\omega \otimes \xi] \in H_{dR}^n(N \otimes M)$. The $\hat{\otimes}$ operation is the standard tensor product in which the wedge product becomes modified by the grading to give $(x \hat{\otimes} y) \wedge (w \hat{\otimes} z) = (-1)^{nm} (x \wedge w) \hat{\otimes} (y \wedge z)$, where $y \in H_{dR}^n(M)$ and $w \in H_{dR}^m(N)$.

In addition, if there are differentiable algebra maps $\phi : N \rightarrow N'$ and $\psi : M \rightarrow M'$, then in terms of the isomorphism above we have $\phi_* \hat{\otimes} \psi_* : H_{dR}^r(N) \hat{\otimes} H_{dR}^s(M) \rightarrow H_{dR}^r(N') \hat{\otimes} H_{dR}^s(M')$ corresponding to $(\phi \otimes \psi)_* : H_{dR}^n(N \otimes M) \rightarrow H_{dR}^n(N' \otimes M')$.

Corollary 2.6 *The graded Hopf algebra $H_{dR}^*(P)$ coacts on $H_{dR}^*(M)$ by*

$$H_{dR}^n(P) \xrightarrow{\lambda_*} H_{dR}^n(P \otimes M) \cong \bigoplus_{n \geq r \geq 0} H_{dR}^r(P) \hat{\otimes} H_{dR}^{n-r}(M).$$

3 Integrals and invariant forms

Again we suppose that P is a Hopf algebra with a bicovariant differential calculus, and that P left coacts on an algebra M (with a given differential structure) by a differentiable map $\lambda : M \rightarrow P \otimes M$. Recall the definition of a normalised left integral on a Hopf algebra.

Definition 3.1 A *left integral* on a Hopf algebra P is a linear map $\int : P \rightarrow k$ such that $(\int \otimes \text{id})\Delta = I_P \cdot \int : P \rightarrow P$. A left integral is said to be *normalised* provided $\int I_P = 1$.

Throughout this section we suppose that P has a normalised left integral \int . Since we are working over a field this is equivalent to assuming that P is a *cosemisimple Hopf algebra* (i.e., a sum of simple coalgebras) [21, 14.0.3]. Given a left P -comodule E with coaction $\bar{\lambda}$, we define a map

$$\mathbb{I} = (\int \otimes \text{id})\bar{\lambda} : E \rightarrow {}^{coP}E := \{\omega \in E \mid \bar{\lambda}(\omega) = 1_P \otimes \omega\}.$$

Lemma 3.2 *The left invariant forms ${}^{coP}(\Omega^n M) := \{\omega \in \Omega^n M \mid \bar{\lambda}(\omega) = 1_P \otimes \omega\}$ form a cochain complex with the usual de Rham differential. Furthermore, the map $\mathbb{I} : \Omega^n M \rightarrow {}^{coP}(\Omega^n M)$ is a cochain map.*

Proof As $d : \Omega^n M \rightarrow \Omega^{n+1} M$ is a left P -comodule map it follows that d preserves the invariant forms. Also we find that

$$\mathbb{I}(d\omega) = (\int \otimes \text{id})\bar{\lambda}(d\omega) = (\int \otimes \text{id})(\text{id} \otimes d)\bar{\lambda}(\omega) = d(\mathbb{I}(\omega)) . \quad \square$$

Now we have two cochain maps, $\mathbb{I} : \Omega^n M \rightarrow {}^{coP}(\Omega^n M)$ and the inclusion map $i : {}^{coP}(\Omega^n M) \rightarrow \Omega^n M$. As the integral is normalised it follows that $\mathbb{I} \circ i : {}^{coP}(\Omega^n M) \rightarrow {}^{coP}(\Omega^n M)$ is the identity. Next consider the induced cohomology maps $H(\mathbb{I}) : H_{dR}^n(M) \rightarrow H^n({}^{coP}(\Omega^* M), d)$ and $H(i) : H^n({}^{coP}(\Omega^* M), d) \rightarrow H_{dR}^n(M)$. We see that $H(\mathbb{I}) \circ H(i)$ is the identity on $H^n({}^{coP}(\Omega^* M), d)$, and as a result $H(i) \circ H(\mathbb{I})$ is a projection on $H_{dR}^n(M)$. It is obvious that the image of $H(i)$ is contained in ${}^{coP}(H_{dR}^n(M))$, but it is less obvious that the image is ${}^{coP}(H_{dR}^n(M))$.

Proposition 3.3 *The image of $H(i) : H^n({}^{coP}(\Omega^* M), d) \rightarrow H_{dR}^n(M)$ is ${}^{coP}(H_{dR}^n(M))$.*

Proof Suppose that $[\omega] \in H_{dR}^n(M)$ and that $(\text{id} \otimes [\bullet])\bar{\lambda}(\omega) = 1_P \otimes [\omega]$. Write $\bar{\lambda}(\omega) = \sum_i p_i \otimes \tau_i$, where $d\tau_i = 0$. Then $\sum_i p_i \otimes [\tau_i] = 1_P \otimes [\omega]$, or

$$\sum_i p_i \otimes \tau_i = 1_P \otimes \omega + \sum_j q_j \otimes d\eta_j .$$

Applying $\int \otimes \text{id}$ to this we get

$$\mathbb{I}(\omega) = \sum_i \int (p_i) \tau_i = \omega + \sum_j \int (q_j) d\eta_j ,$$

showing that $[\mathbb{I}(\omega)] = [\omega] \in H_{dR}^n(M)$, and $[\mathbb{I}(\omega)]$ is in the image of $H(i)$. \square

The above discussion leads to the main result of this section.

Theorem 3.4 *Let P be a Hopf algebra with a bicovariant differential calculus. Suppose that P coacts on M by a differentiable left coaction.*

- 1) *If P has a normalised left integral, then there is an isomorphism $H(i) : H^n({}^{coP}(\Omega^* M), d) \rightarrow {}^{coP}(H_{dR}^n(M))$.*
- 2) *If P is connected then ${}^{coP}(H_{dR}^n(M)) = H_{dR}^n(M)$.*
- 3) *If P is a connected cosemisimple Hopf algebra, then ${}^{coP}(H_{dR}^n(M)) = H_{dR}^n(M) = H^n({}^{coP}(\Omega^* M), d)$.*

Corollary 3.5 *If P is a Hopf algebra with bicovariant differential structure, and P has a normalised left integral, then the inclusion map induces an isomorphism $H(i) : H^n({}^{coP}(\Omega^*P), d) \rightarrow {}^{coP}(H^n_{dR}(P))$. If P is connected then ${}^{coP}(H^n_{dR}(P)) = H^n_{dR}(P)$.*

Remark 3.6 In [14, Theorem 3.1] the cohomology of Hopf algebras $SL_q(N)$, $SO_q(N)$ and $Sp_q(N)$ was shown by explicit calculation to be the same as the cohomology of their left invariant forms for certain bicovariant differential calculi. All these Hopf algebras are cosemisimple, i.e., they have a normalised left integral, and they can be shown to be connected, hence this part of [14, Theorem 3.1] follows from Corollary 3.5. On the other hand it is also computed in [14, Theorem 3.1] that for $O_q(2n+1)$ and certain bicovariant differential calculi, the de Rham cohomology decomposes into a sum of two copies of the de Rham cohomology of invariant forms. Although $O_q(2n+1)$ are cosemisimple Hopf algebras, they are not connected with respect to differential calculi discussed in [14]. In addition to the de Rham cohomology class 1_P there is another non-trivial class induced by the quantum determinant. Although [14, Theorem 3.1] cannot be inferred from Theorem 3.4 and Corollary 3.5, the latter indicate the origin of this decomposition: every ‘connected component’ leads to contributions to the de Rham cohomology that go beyond the invariant part. Similarly, the decomposition of the de Rham cohomology of $GL_q(N)$ in [14, Theorem 3.1(2)] can be expected from Theorem 3.4 in view of the fact that the powers of the quantum determinant induce nontrivial cohomology classes in $H^0_{dR}(GL_q(N))$.

4 Hopf cochain cohomology

In this section P is a Hopf algebra.

Definition 4.1 Suppose that F is a left P -comodule. Define $D^n = P^{\otimes n+1} \otimes F$ for $n \geq 0$, with the tensor product left coaction. The map $d : D^n \rightarrow D^{n+1}$ is defined by

$$d(p_0 \otimes \dots \otimes p_n \otimes f) = \sum_{n+1 \geq i \geq 0} (-1)^i p_0 \otimes \dots \otimes p_{i-1} \otimes 1 \otimes p_i \otimes \dots \otimes p_n \otimes f.$$

It follows that d is left P -covariant, and that $d^2 = 0$. The cohomology of the P -invariant complex $({}^{coP}D^n, d)$ is called a *Hopf cochain cohomology of P with coefficients in F* and is denoted by $H^n_c(P; F)$.

The reader familiar with the cohomology theory of algebras (and with the descent theory in particular) will recognise in (D^n, d) in Definition 4.1 the *Amitsur complex* of the algebra P [1] (cf. [2, Section 6]). Note, however, that, motivated by the group cohomology, for a Hopf cochain cohomology we take the coinvariant part of the Amitsur complex.

Proposition 4.2 *Let F be a left P -comodule with coaction $\lambda : F \rightarrow P \otimes F$. Suppose that there is a left action $\mu : P \otimes F \rightarrow F$ which is a left P -comodule map with respect to λ and the tensor product coaction, i.e., such that, for all $p \in P$ and $f \in F$, $\lambda(\mu(p \otimes f)) = p_{(1)}f_{[-1]} \otimes \mu(p_{(2)} \otimes f_{[0]})$. Then $H^n_c(P; F) = 0$ for $n \geq 1$ and $H^0_c(P; F) \cong {}^{coP}F$, with the isomorphism mapping $f \in {}^{coP}F$ to $[1_P \otimes f] \in H^0_c(P; F)$.*

Proof Define the map $h : P^{\otimes n+2} \otimes F = D^{n+1} \rightarrow P^{\otimes n+1} \otimes F = D^n$ for $n \geq -1$ by

$$h(p_0 \otimes \dots \otimes p_{n+1} \otimes f) = (-1)^{n+1} p_0 \otimes \dots \otimes p_n \otimes p_{n+1} \cdot f .$$

Now we calculate

$$\begin{aligned} h d(p_0 \otimes \dots \otimes p_{n+1} \otimes f) &= \sum_{n+1 \geq i \geq 0} (-1)^{i+n} p_0 \otimes \dots \otimes p_{i-1} \otimes 1 \otimes p_i \otimes \dots \otimes p_{n+1} \cdot f \\ &\quad + p_0 \otimes \dots \otimes p_{n+1} \otimes f , \\ dh(p_0 \otimes \dots \otimes p_{n+1} \otimes f) &= (-1)^{n+1} 1 \otimes p_0 \otimes \dots \otimes p_n \otimes p_{n+1} \cdot f + \dots \\ &\quad + p_0 \otimes \dots \otimes p_n \otimes 1 \otimes p_{n+1} \cdot f . \end{aligned}$$

From this we see that

$$(d \circ h + h \circ d)(p_0 \otimes \dots \otimes p_{n+1} \otimes f) = p_0 \otimes \dots \otimes p_{n+1} \otimes f .$$

Since h is left P -covariant, we have a cochain homotopy which contracts the complex, showing that $H_c^n(P; F) = 0$ for $n \geq 1$. To find $H_c^0(P; F)$ we need to be more careful, as our hypothetical D^{-1} is not part of the complex. However we can still write $x = dh(x) + h(dx)$ for $x \in D^0$. Then if $x \in {}^{coP}D^0$ and $dx = 0$, we have $x = 1 \otimes h(x)$, so we can identify $\ker d : D^0 \rightarrow D^1$ as $1_P \otimes {}^{coP}F$. \square

A left F -comodule P that satisfies assumptions of Proposition 4.2, i.e., such that it is also a left P -module with an action that is compatible with a coaction is known as a left *Hopf module* (cf. [21]). As an explicit example one can consider *cleft extensions* of algebras (cf. [5], [12]).

Example 4.3 Let F be a left P -comodule algebra with the coaction λ , and let $M = {}^{coP}F$ be the algebra of coinvariants. Suppose there exists a left P -colinear map $\Phi : P \rightarrow F$ such that $\Phi(1) = 1$ and Φ is convolution-invertible, i.e., there is a map $\Phi^{-1} : P \rightarrow F$ such that for all $p \in P$, $\Phi(p_{(1)})\Phi^{-1}(p_{(2)}) = \Phi^{-1}(p_{(1)})\Phi(p_{(2)}) = \epsilon(p)1_P$. Then F is called a *cleft extension of M* . This is an example of a Hopf-Galois extension that has a geometric meaning of a trivial principal bundle (cf. [9]). One can show that $F \cong P \otimes M$ as a right M -module and as a left P -comodule, where the coaction in $P \otimes M$ is given by $\Delta \otimes \text{id}$ (cf. [12, Theorem 9]). Explicitly, the isomorphism $\Theta : F \rightarrow P \otimes M$ and its inverse are

$$\Theta(f) = f_{[-2]} \otimes \Phi^{-1}(f_{[-1]})f_{[0]}, \quad \Theta^{-1}(p \otimes x) = \Phi(p)x.$$

The map Θ is an isomorphism of algebras provided the product in $P \otimes M$ is given by the formula

$$(p \otimes x)(q \otimes y) = p_{(1)}q_{(1)} \otimes \Phi^{-1}(p_{(2)}q_{(2)})\Phi(p_{(3)})x\Phi(q_{(3)})y,$$

(cf. [12, Theorem 11]).

In this case there is a left action of P on F , $\mu : P \otimes F \rightarrow F$ given by $p \otimes f \mapsto \Phi(pf_{[-2]})\Phi^{-1}(f_{[-1]})f_{[0]}$. When F is viewed as $P \otimes M$ this is simply the multiplication of elements in P , hence it is a left action. That fact that μ is compatible with the coactions is clear once one identifies F with $P \otimes M$ via Θ . To check this explicitly, take any $f \in F$

and $p \in P$ and compute

$$\begin{aligned}
\lambda(\mu(p \otimes f)) &= \lambda(\Phi(pf_{[-2]})\Phi^{-1}(f_{[-1]})f_{(0)}) \\
&= p_{(1)}f_{[-5]}(Sf_{[-2]})f_{[-1]} \otimes \Phi(p_{(2)}f_{[-4]})\Phi^{-1}(f_{[-3]})f_{(0)} \\
&= p_{(1)}f_{[-2]} \otimes \Phi(pf_{[-2]})\Phi^{-1}(f_{[-1]})f_{[0]} \\
&= p_{(1)}f_{[-1]} \otimes \mu(p_{(2)} \otimes f_{[0]}),
\end{aligned}$$

as required. Thus if F is a cleft extension of M , then $H_c^n(P; F) = 0$ for $n \geq 1$ and $H_c^0(P; F) \cong {}^{coP}F = M$.

For another example of a left P -comodule which is a left P -Hopf module, so that the assumptions of Proposition 4.2 hold, take F to be equal a P -bimodule $\Omega^n P$ of differential n -forms on P in a left covariant differential structure (cf. Corollary 6.4 below).

It will be convenient later to have an explicit description of this cohomology theory which does not involve P -invariance, so we give an alternative formulation.

Proposition 4.4 *For a left P -comodule F with coaction $\lambda(f) = f_{[-1]} \otimes f_{[0]}$, define a cochain complex (G^*, \bar{d}) by $G^n = P^n \otimes F$ for $n \geq 0$ with derivation $\bar{d}f = 1_P \otimes f - \lambda(f)$ and*

$$\begin{aligned}
\bar{d}(p_1 \otimes \dots \otimes p_n \otimes f) &= 1_P \otimes p_1 \otimes \dots \otimes p_n \otimes f - \Delta(p_1) \otimes \dots \otimes p_n \otimes f + \dots \\
&\quad + (-1)^n p_1 \otimes \dots \otimes \Delta(p_n) \otimes f - (-1)^n p_1 \otimes \dots \otimes p_n \otimes \lambda(f).
\end{aligned}$$

Then there is a cochain isomorphism $\theta : (G^, \bar{d}) \rightarrow ({}^{coP}D^*, d)$ given by $\theta(f) = S(f_{[-1]}) \otimes f_{[0]}$ and*

$$\begin{aligned}
\theta(p_1 \otimes \dots \otimes p_n \otimes f) &= S(p_{1(1)}) \otimes p_{1(2)} S(p_{2(1)}) \otimes \dots \otimes p_{n-1(2)} S(p_{n(1)}) \\
&\quad \otimes p_{n(2)} S(f_{[-1]}) \otimes f_{[0]}.
\end{aligned}$$

Proof By explicit calculation the image of θ is in the P -invariant part of D^n . To show that θ is a cochain map we calculate

$$\begin{aligned}
\theta(1_P \otimes p_1 \otimes \dots \otimes p_n \otimes f) &= 1_P \otimes S(p_{1(1)}) \otimes p_{1(2)} S(p_{2(1)}) \otimes \dots \otimes p_{n(2)} S(f_{[-1]}) \otimes f_{[0]}, \\
\theta(\Delta(p_1) \otimes \dots \otimes p_n \otimes f) &= S(p_{1(1)}) \otimes 1_P \otimes p_{1(2)} S(p_{2(1)}) \otimes \dots \otimes p_{n(2)} S(f_{[-1]}) \otimes f_{[0]}, \\
\theta(p_1 \otimes \dots \otimes \Delta(p_n) \otimes f) &= S(p_{1(1)}) \otimes p_{1(2)} S(p_{2(1)}) \otimes \dots \otimes 1_P \otimes p_{n(2)} S(f_{[-1]}) \otimes f_{[0]}, \\
\theta(p_1 \otimes \dots \otimes p_n \otimes \lambda(f)) &= S(p_{1(1)}) \otimes p_{1(2)} S(p_{2(1)}) \otimes \dots \otimes p_{n(2)} S(f_{[-1]}) \otimes 1_P \otimes f_{[0]}.
\end{aligned}$$

Adding these terms with the appropriate signs shows that $\theta \circ \bar{d} = d \circ \theta$. The inverse function is given by

$$\begin{aligned}
\theta^{-1}(p_0 \otimes f) &= \epsilon(p_0) f, \\
\theta^{-1}(p_0 \otimes p_1 \otimes f) &= \epsilon(p_0) p_1 f_{[-1]} \otimes f_{[0]}, \\
\theta^{-1}(p_0 \otimes p_1 \otimes p_2 \otimes f) &= \epsilon(p_0) p_1 p_{2(1)} f_{[-2]} \otimes p_{2(2)} f_{[-1]} \otimes f_{[0]}, \\
\theta^{-1}(p_0 \otimes p_1 \otimes p_2 \otimes p_3 \otimes f) &= \epsilon(p_0) p_1 p_{2(1)} p_{3(1)} f_{[-3]} \otimes p_{2(2)} p_{3(2)} f_{[-2]} \otimes p_{3(3)} f_{[-1]} \otimes f_{[0]},
\end{aligned}$$

and so on. \square

5 Spectral sequences

This section contains well known material on spectral sequences and double complexes, which we have taken from [19], though we have slightly specialised the results.

Definition 5.1 A double complex of bidegree (r, t) is a collection of vector spaces $E^{n,m}$ (taken to be zero if either $n < 0$ or $m < 0$) and linear maps $D : E^{n,m} \rightarrow E^{n+r, m+t}$ with $D \circ D = 0$. A spectral sequence of degree $\geq s$ is a collection $(E_r^{*,*}, D_r)$ of double complexes of bidegree $(r, 1-r)$ for $r \geq s$ so that

$$E_{r+1}^{n,m} \cong H^{n,m}(E_r, D_r) = (\ker D_r : E_r^{n,m} \rightarrow E_r^{n+r, m+1-r}) / (\text{im } D_r : E_r^{n-r, m+r-1} \rightarrow E_r^{n,m}) .$$

The problem is that knowledge of D_r does not necessarily imply knowledge of D_{r+1} , though additional information, such as a ring structure, may help in specific cases. The name of the game is to be able to calculate the limit $\lim_{r \rightarrow \infty} E_r^{n,m}$. The conditions that we put on the vanishing of $E_r^{n,m}$ ensure that the $E_r^{n,m}$ stabilise for sufficiently large r .

Next we give an example where spectral sequences arise, the example that is essential for the van Est spectral sequence.

Example 5.2 Start with a double complex $C^{n,m}$ (taken to be zero if either $n < 0$ or $m < 0$) with two differentials, d' of bidegree $(1, 0)$ and d'' of bidegree $(0, 1)$, which satisfy $d'' \circ d' + d' \circ d'' = 0$. The total complex T^s is defined as $T^s = \bigoplus_i C^{i, s-i}$ with differential $d = d' + d''$. Define

$$\begin{aligned} H_I^{n,m}(C) &= H^{n,m}(C, d') = (\ker d' : C^{n,m} \rightarrow C^{n+1,m}) / (\text{im } d' : C^{n-1,m} \rightarrow C^{n,m}) , \\ H_{II}^{n,m}(C) &= H^{n,m}(C, d'') = (\ker d'' : C^{n,m} \rightarrow C^{n,m+1}) / (\text{im } d'' : C^{n,m-1} \rightarrow C^{n,m}) . \end{aligned}$$

The map d'' induces a differential \bar{d}'' on the bigraded complex $H_I^{n,m}(C)$ by $\bar{d}''[c] = [d''c]$, and d' induces a differential \bar{d}' on the double complex $H_{II}^{n,m}(C)$ by $\bar{d}'[c] = [d'c]$, where $[c]$ denotes the equivalence class of $c \in C^{n,m}$ under the quotient. Now there are two spectral sequences $({}_I E_r^{*,*}, {}_I D_r)$ and $({}_{II} E_r^{*,*}, {}_{II} D_r)$ for $r \geq 2$ with ${}_I E_2^{n,m} \cong H^{n,m}(H_I^{*,*}(C), \bar{d}')$ and ${}_{II} E_2^{n,m} \cong H^{n,m}(H_{II}^{*,*}(C), \bar{d}'')$ which both converge to the cohomology of the total complex $H^*(T, d)$. This means that $\bigoplus_i E_\infty^{i, n-i} = H^n(T, d)$, denoting the limits by $E_\infty^{n,m}$.

6 The van Est spectral sequence

Again, P is a Hopf algebra. The construction of the van Est spectral sequence for Hopf algebras is based on the following example of a double complex.

Example 6.1 Suppose that there are left P -comodules F^n for $n \geq 0$, and left P -comodule maps $\bar{d} : F^n \rightarrow F^{n+1}$ with the property that $\bar{d} \circ \bar{d} = 0$. Construct a double complex $C^{n,m} = P^{\otimes n} \otimes F^m$ ($n, m \geq 0$) with differentials $d' : P^{\otimes n} \otimes F^m \rightarrow P^{\otimes n+1} \otimes F^m$ being the Hopf-cochain differentials given in Proposition 4.4 and $d'' : P^{\otimes n} \otimes F^m \rightarrow P^{\otimes n} \otimes F^{m+1}$ being $(-1)^n \text{id} \otimes \bar{d}$. Note that the $(-1)^n$ factor is included to force the condition $d'' \circ d' + d' \circ d'' = 0$.

Lemma 6.2 *Suppose that the cochain complex (F^n, \bar{d}) in Example 6.1 satisfies the additional condition that each F^n has a left P -action : $P \otimes F^n \rightarrow F^n$ that is a left P -comodule map, i.e., each of the F^n is a left P -Hopf module. Then the spectral sequence ${}_I E_r^{n,m}$ described in Example 5.2 converges to $H^s({}^{coP}F, \bar{d})$.*

Proof From Proposition 4.2, $H_I^{n,m}(C) = H^{n,m}(C, d') = 0$ for $n > 0$ and $H_I^{0,m}(C) \cong {}^{coP}F^m$, with the isomorphism mapping $f \in {}^{coP}F^m$ to $[1_P \otimes f] \in H_I^{0,m}(C)$. By definition $\bar{d}''[1_P \otimes f] = [1_P \otimes \bar{d}f]$, so the isomorphism identifies the complexes $(H_I^{0,m}(C), \bar{d}'')$ and $({}^{coP}F^m, \bar{d})$. Thus we have ${}_I E_2^{n,m} = 0$ for $n > 0$ and ${}_I E_2^{0,m} = H^m({}^{coP}F, \bar{d})$. But now every map ${}_I D_r$ ($r \geq 2$) must be zero, as it either maps into or out of zero. This means that the ${}_I E_r^{*,*}$ spectral sequence stabilises at $r = 2$, giving the result. \square

Theorem 6.3 *Suppose that P is a Hopf algebra, and that there is a cochain complex (F^n, \bar{d}) of left P -comodules with the differential \bar{d} being a comodule map. Additionally suppose that each F^n has a left P -action : $P \otimes F^n \rightarrow F^n$ which is a left P -comodule map. Then there is a spectral sequence beginning with $E_2^{n,m} = H_c^n(P; H^m(F^*, \bar{d}))$ which converges to $H^s({}^{coP}F, \bar{d})$.*

Proof We identify the spectral sequence ${}_I E_2^{n,m}$ in Example 6.1. First $\ker(\text{id}_{P^{\otimes n}} \otimes \bar{d}) = P^{\otimes n} \otimes \ker \bar{d}$ and $\text{im}(\text{id}_{P^{\otimes n}} \otimes \bar{d}) = P^{\otimes n} \otimes \text{im} \bar{d}$. The short exact sequence

$$0 \longrightarrow (\text{im } \bar{d} : F^{m-1} \rightarrow F^m) \longrightarrow (\ker \bar{d} : F^m \rightarrow F^{m+1}) \longrightarrow H^m(F^*, \bar{d}) \longrightarrow 0$$

remains exact if we tensor on the left with $P^{\otimes n}$. Putting these results together,

$$H^m(P^{\otimes n} \otimes F^*, \text{id} \otimes \bar{d}) = P^{\otimes n} \otimes H^m(F^*, \bar{d}).$$

Applying the induced differential to this gives the result. \square

Corollary 6.4 *Suppose that P is a Hopf algebra with bicovariant differential calculus. Then there is a spectral sequence beginning with $E_2^{n,m} = H_c^n(P; H_{dR}^m(P))$ which converges to $H^s({}^{coP}(\Omega^*P), d)$.*

Proof Put (F^*, \bar{d}) equal to the de Rham complex (Ω^*P, d) in Theorem 6.3. The action of left multiplication : $P \otimes \Omega^n P \rightarrow \Omega^n P$ is a left P -comodule map. \square

Example 6.5 Suppose that F is a cleft extension of M as described in Example 4.3. We thus know that F is a left P -Hopf module. If, in addition, $\Phi : P \rightarrow F$ is an algebra map, then the left action μ of P on F is induced from the product in F via the map Φ , i.e., $\mu(p \otimes f) = \Phi(p)f$. In this case, for any left P -covariant differential structure ΩF , the F -bimodule of n -forms $\Omega^n F$ is a left P -module via the map Φ , i.e., there are actions $\mu_n : P \otimes \Omega^n F \rightarrow \Omega^n F$, given by $p \otimes \omega = \Phi(p) \cdot \omega$. Since ΩF is a P -covariant calculus, we can compute, for any $p \in P$ and $\omega \in \Omega^n F$,

$$\lambda^n(\mu_n(p \otimes \omega)) = \lambda^n(\Phi(p)\omega) = p_{(1)}\omega_{[-1]} \otimes \Phi(p_{(2)})\omega_{[0]} = p_{(1)}\omega_{[-1]} \otimes \mu_n(p_{(2)} \otimes \omega),$$

where $\lambda^n : \Omega^n F \rightarrow P \otimes \Omega^n F$ is the P -coaction. Thus each of the $\Omega^n F$ is a left Hopf P -module, and we can infer from Theorem 6.3 that there is a spectral sequence beginning with $E_2^{n,m} = H_c^n(P; H^m(\Omega F, \bar{d}))$ that converges to $H^s(M, \bar{d})$. General constructions and explicit examples of differential structure on cleft extensions can be found in [10] (beware that [10] uses right coactions rather than left coactions).

7 Braidings and n -forms

The material contained in this section has existed for a long time, and was mostly developed by Woronowicz in [22]. Thus we merely give a brief description and point out that proofs can be seen in [22].

For a Hopf algebra P with a bicovariant differential structure, there are right and left coactions ρ and λ , given by

$$\rho(p.dq) = p_{(1)}.dq_{(1)} \otimes p_{(2)} q_{(2)} , \quad \lambda(p.dq) = p_{(1)} q_{(1)} \otimes p_{(2)}.dq_{(2)} ,$$

for all $p, q \in P$. We shall use the index notation $\rho(\xi) = \xi_{[0]} \otimes \xi_{[1]}$ and $\lambda(\xi) = \xi_{[-1]} \otimes \xi_{[0]}$ (summation understood). $\Omega^1 P$ is a bicomodule with these coactions, meaning that the left and right coactions commute, i.e., we can write the following without ambiguity:

$$(\lambda \otimes \text{id})\rho(\xi) = (\text{id} \otimes \rho)\lambda(\xi) = \xi_{[-1]} \otimes \xi_{[0]} \otimes \xi_{[1]} .$$

Denote by L^1 the left invariant 1-forms ${}^{\text{co}P}(\Omega^1 P)$. There is a map $Y : \Omega^1 P \rightarrow P \otimes L^1$ defined by $Y(\xi) = \xi_{[-2]} \otimes S(\xi_{[-1]}) \cdot \xi_{[0]}$. In terms of P -actions and coactions we get $Y : \bullet\Omega^1 P\bullet \rightarrow \bullet P\bullet \otimes (L^1)\bullet$. The upper (resp. lower) dots indicate what coaction (resp. action) we take, i.e., the right action and coaction are the tensor product ones, whereas the left action and coaction are purely on the first component. The right action on L^1 is $\chi \triangleleft p = S(p(1)) \cdot \chi \cdot p(2)$. Note that as a right P -module and a right P -comodule L^1 satisfies the Yetter-Drinfeld condition. The main consequences of this are explained in the following remarks.

Remark 7.1 A tensor category consisting of objects which are right P -modules and right P -comodules, with morphisms which are right P -module and right P -comodule maps, is said to satisfy the (right) *Yetter-Drinfeld condition* if $\rho(\eta \triangleleft a) = \eta_{[0]} \triangleleft a_{(2)} \otimes S(a_{(1)}) \eta_{[1]} a_{(3)}$. Here η is any element of any object V and a is an element of P . The symbol \triangleleft indicates the right action of P on V and $\eta_{[0]} \otimes \eta_{[1]}$ denotes the right coaction. The tensor product has the usual tensor product action and coaction, and the associator is trivial. Objects of a category that satisfies the (right) Yetter-Drinfeld condition are known as *Yetter-Drinfeld* or *crossed* modules. The map $\sigma_{VW} : V \otimes W \rightarrow W \otimes V$ defined by $\sigma(\xi \otimes \eta) = \eta_{[0]} \otimes \xi \triangleleft \eta_{[1]}$ is a braiding for the category. If the antipode S is invertible, there is an inverse braiding $\sigma^{-1}(\eta \otimes \xi) = \xi \triangleleft S^{-1}(\eta_{[1]}) \otimes \eta_{[0]}$.

We call an element $x \in V \otimes V$ *symmetric* provided $\sigma(x) = x$, and define $V \wedge V$ to be the quotient of $V \otimes V$ by the subspace $\mathcal{S}(V)$ of symmetric elements. In the same manner we can define $V \wedge V \wedge V$ as the quotient of $V \otimes V \otimes V$ by the subspace generated by $\mathcal{S}(V) \otimes V$ and $V \otimes \mathcal{S}(V)$, and so on.

Remark 7.2 We shall take the isomorphism Y mentioned earlier seriously and *define* $\bullet\Omega^1 P\bullet = \bullet P\bullet \otimes (L^1)\bullet$. This naturally leads to the definitions $\bullet\Omega^2 P\bullet = \bullet P\bullet \otimes (L^1)\bullet \wedge (L^1)\bullet$, $\bullet\Omega^3 P\bullet = \bullet P\bullet \otimes (L^1)\bullet \wedge (L^1)\bullet \wedge (L^1)\bullet$ etc. The wedge product making $\Omega^* P$ into a graded algebra is $(p \otimes v) \wedge (q \otimes w) = p q_{(1)} \otimes v \triangleleft q_{(2)} \wedge w$, for all $p, q \in P$ and $v, w \in L^1$. The de Rham differential on the left invariant 1-forms is given by

$$d(S(p_{(1)}).dp_{(2)}) = dS(p_{(1)}) \wedge dp_{(2)} = -S(p_{(1)}).dp_{(2)} \wedge S(p_{(3)}).dp_{(4)} ,$$

and by explicit calculation this $d : L^1 \rightarrow L^1 \wedge L^1$ is a right comodule map. To derive this equation we used the result

$$0 = d(\epsilon(p)) = d(S(p_{(1)})p_{(2)}) = d(S(p_{(1)})) \cdot p_{(2)} - S(p_{(1)}) \cdot d(p_{(2)}),$$

so that $d(S(p)) = -S(p_{(1)}) \cdot dp_{(2)} \cdot S(p_{(3)})$. The differential is extended to $d : (L^1)^{\wedge n} \rightarrow (L^1)^{\wedge n+1}$ by

$$\begin{aligned} d(\xi_1 \wedge \xi_2 \wedge \dots \wedge \xi_n) &= d\xi_1 \wedge \xi_2 \wedge \dots \wedge \xi_n - \xi_1 \wedge d\xi_2 \wedge \dots \wedge \xi_n \\ &\quad + (-1)^{n+1} \xi_1 \wedge \xi_2 \wedge \dots \wedge d\xi_n, \end{aligned}$$

and to $d : \Omega^n P \rightarrow \Omega^{n+1} P$ by $d(p \otimes v) = p_{(1)} \otimes S(p_{(2)}) \cdot dp_{(3)} \wedge v + p \otimes dv$.

Remark 7.3 The discussion of the preceding remarks allows us in principle to calculate the cohomology of the left invariant forms $(L^1)^{\wedge n}$, given the (often finite dimensional) right P module and comodule L^1 and $d : L^1 \rightarrow L^1 \wedge L^1$. However this construction does not use anything corresponding to the Lie algebra of a Lie group.

Incidentally, though the braiding introduced here may seem rather arbitrary, it is not too difficult to justify. One way is to see that it is the braiding (in the sense of [17]) corresponding to the left covariant derivative on the bimodule $\Omega^1 P$ which kills all left invariant forms. There will be a better reason later.

8 Adjoint coactions and the Hopf-Lie algebra

In the case of a Lie group G , the left Adjoint action $: G \times G \rightarrow G$ is given by $\text{Ad}_g(h) = ghg^{-1}$. Differentiating this in the second variable h in the direction of v in the Lie algebra \mathfrak{g} , we get $\text{Ad}_{g*}(v) = gvg^{-1} \in \mathfrak{g}$. Finally differentiating with respect to g in the direction $w \in \mathfrak{g}$ we get the Lie bracket $[w, v] = wv - vw$. By following this prescription for a Hopf algebra we get the most direct justification for the braiding in section 7.

Again P is a Hopf algebra with a bicovariant differential calculus $\Omega^* P$.

Definition 8.1 For a Hopf algebra P , the *Hopf-Lie algebra* is defined as

$$\mathfrak{p} = \{ \psi : \Omega^1 P \rightarrow k : \psi(\xi \cdot p) = \psi(\xi) \epsilon(p) \quad \forall p \in P \}.$$

The left adjoint P -coaction on P is defined by $\text{Ad}^L(p) = p_{(1)} S(p_{(3)}) \otimes p_{(2)}$. With the help of this coaction we define a *bracket*, for all $\alpha, \beta \in \mathfrak{p}$,

$$[\alpha, \beta] = \alpha \circ d \circ (\text{id} \otimes \beta) \circ \Pi_0 \circ \text{Ad}_*^L : \Omega^1 P \rightarrow k.$$

This definition of a Hopf-Lie algebra has a classical motivation. Classically, the Lie algebra of a Lie group can be identified with a space dual to the cotangent space. In the case of a general Hopf algebra P , k is a right P -module with the action given by the counit ϵ (remember that ϵ is a character of P). Also $\Omega^1 P$ is a right P -module, and so \mathfrak{p} is simply a space of right P -linear maps $\Omega^1 P \rightarrow k$.

Proposition 8.2 [22] *The Hopf-Lie algebra is closed under the bracket $[\bullet, \bullet]$, i.e., for all $\alpha, \beta \in \mathfrak{p}$, $[\alpha, \beta] : \Omega^1 P \rightarrow k$ is in \mathfrak{p} . Furthermore*

$$[\alpha, \beta](\xi) = \alpha(d(\xi_{[-1]})) \beta(\xi_{[0]}) - \alpha(\xi_{[-1]} S(\xi_{[1]}) \cdot d(\xi_{[2]})) \beta(\xi_{[0]}).$$

Proof To show that $[\alpha, \beta] \in \mathfrak{p}$, using the fact that $\beta \in \mathfrak{p}$, we write

$$\begin{aligned}
(\text{id} \otimes \beta) \circ \Pi_2 \circ \text{Ad}_*^L(\xi.p) &= (\text{id} \otimes \beta)(\xi_{[-1]} p_{(1)} S(p_{(3)}) S(\xi_{[1]}) \otimes \xi_{[0]} p_{(2)}) \\
&= \xi_{[-1]} p_{(1)} S(p_{(3)}) S(\xi_{[1]}) \beta(\xi_{[0]} p_{(2)}) \\
&= \xi_{[-1]} p_{(1)} S(p_{(3)}) S(\xi_{[1]}) \beta(\xi_{[0]}) \epsilon(p_{(2)}) \\
&= \xi_{[-1]} p_{(1)} S(p_{(2)}) S(\xi_{[1]}) \beta(\xi_{[0]}) \\
&= \xi_{[-1]} S(\xi_{[1]}) \beta(\xi_{[0]}) \epsilon(p) .
\end{aligned} \tag{1}$$

Next we have, using the fact that d is a derivation,

$$\begin{aligned}
[\alpha, \beta](\xi) &= \alpha(d(\xi_{[-1]} S(\xi_{[1]}))) \beta(\xi_{[0]}) \\
&= \alpha(d(\xi_{[-1]}) . S(\xi_{[1]})) \beta(\xi_{[0]}) + \alpha(\xi_{[-1]} . d(S(\xi_{[1]}))) \beta(\xi_{[0]}) \\
&= \alpha(d(\xi_{[-1]})) \epsilon(S(\xi_{[1]})) \beta(\xi_{[0]}) + \alpha(\xi_{[-1]} . d(S(\xi_{[1]}))) \epsilon(\xi_{[2]}) \beta(\xi_{[0]}) \\
&= \alpha(d(\xi_{[-1]})) \beta(\xi_{[0]}) + \alpha(\xi_{[-1]} . d(S(\xi_{[1]})) . \xi_{[2]}) \beta(\xi_{[0]}) \\
&= \alpha(d(\xi_{[-1]})) \beta(\xi_{[0]}) - \alpha(\xi_{[-1]} S(\xi_{[1]}) . d(\xi_{[2]})) \beta(\xi_{[0]}) . \quad \square
\end{aligned} \tag{2}$$

The next result justifies the definition of d on the left invariant 1-forms given in Remark 7.2, and shows that d on the left invariant 1-forms is dual to the Lie bracket on \mathfrak{p} . The $\text{id} \otimes \text{id} - \sigma$ appearing in the formula allows us to take the quotient from the tensor product to the wedge product.

Proposition 8.3 [22] *For any $p \in P$, let $\xi = S(p_{(1)}) . dp_{(2)}$ be the corresponding left invariant 1-form. Then*

$$\begin{aligned}
[\alpha, \beta](\xi) &= \text{ev}(\text{id} \otimes \text{ev} \otimes \text{id})((\alpha \otimes \beta) \otimes (\text{id} \otimes \text{id} - \sigma)(-S(p_{(1)}) . dp_{(2)} \otimes S(p_{(3)}) . dp_{(4)})) \\
&= \text{ev}(\text{id} \otimes \text{ev} \otimes \text{id})((\alpha \otimes \beta) \otimes (\text{id} \otimes \text{id} - \sigma)(d\xi)).
\end{aligned}$$

Proof This is proven by explicit calculation, beginning from 8.2.

$$\begin{aligned}
[\alpha, \beta](\xi) &= -\alpha(S(\xi_{[1]}) . d(\xi_{[2]})) \beta(\xi_{[0]}) \\
&= -\alpha(S(S(p_{(2)}) p_{(5)}) . d(S(p_{(1)}) p_{(6)})) \beta(S(p_{(3)}) . dp_{(4)}) \\
&= -\alpha(S(p_{(5)}) S^2(p_{(2)}) S(p_{(1)}) . dp_{(6)}) \beta(S(p_{(3)}) . dp_{(4)}) \\
&\quad - \alpha(S(p_{(5)}) S^2(p_{(2)}) . dS(p_{(1)}) . p_{(6)}) \beta(S(p_{(3)}) . dp_{(4)}) \\
&= -\alpha(S(p_{(3)}) . dp_{(4)}) \beta(S(p_{(1)}) . dp_{(2)}) \\
&\quad - \alpha(S(p_{(5)}) S^2(p_{(2)}) . dS(p_{(1)})) \beta(S(p_{(3)}) . dp_{(4)}) \\
&= -\alpha(S(p_{(3)}) . dp_{(4)}) \beta(S(p_{(1)}) . dp_{(2)}) \\
&\quad + \alpha(S(p_{(7)}) S^2(p_{(4)}) S(p_{(1)}) . dp_{(2)} . S(p_{(3)})) \beta(S(p_{(5)}) . dp_{(6)}) \\
&= -\alpha(S(p_{(3)}) . dp_{(4)}) \beta(S(p_{(1)}) . dp_{(2)}) \\
&\quad + \alpha(S(p_{(6)}) S^2(p_{(3)}) S(p_{(1)}) . dp_{(2)}) \beta(S(p_{(4)}) . dp_{(5)}) \\
&= -\alpha(S(p_{(3)}) . dp_{(4)}) \beta(S(p_{(1)}) . dp_{(2)}) \\
&\quad + \alpha((S(p_{(1)}) . dp_{(2)}) \triangleleft S(p_{(3)}) p_{(6)}) \beta(S(p_{(4)}) . dp_{(5)}) \\
&= -\text{ev}(\text{id} \otimes \text{ev} \otimes \text{id})((\alpha \otimes \beta) \otimes (S(p_{(1)}) . dp_{(2)} \otimes S(p_{(3)}) . dp_{(4)} \\
&\quad - S(p_{(4)}) . dp_{(5)} \otimes (S(p_{(1)}) . dp_{(2)}) \triangleleft S(p_{(3)}) p_{(6)})),
\end{aligned}$$

as required for the first equality. The second equality follows from the formula for d in 7.2. \square

9 The Hopf-Lie algebra cohomology

Here we continue from the last section, and ask what the analogue of the Lie algebra cohomology is for a Hopf algebra with bicovariant differential calculus. Without the Jacobi identity the standard formula will not work, we have to include the braiding in the definition of the cochain complex. This section will show how this can be done.

To define the Lie algebra cohomology, we need to transfer the braiding on the left invariant forms to the Lie algebra. There is an evaluation map $\mathfrak{p} \otimes L^1 \rightarrow k$, and this identifies the Lie algebra with the dual of the left invariant forms. Now we look at the dual operation on objects of a Yetter-Drinfeld category.

Remark 9.1 Suppose that the antipode S on P is invertible. Following on from Remark 7.1, we define the dual of an object V in a category with the Yetter-Drinfeld condition to be the vector space dual V^* with right action and coaction

$$(\alpha \lhd p)(v) = \alpha(v \lhd S^{-1}(p)) , \quad \alpha_{[0]}(v) \alpha_{[1]} = \alpha(v_{[0]}) S(v_{[1]}) ,$$

for all $p \in P$, $v \in V$ and $\alpha \in V^*$. The evaluation map $\text{ev} : V^* \otimes V \rightarrow k$ preserves the action and coaction. As the action and coaction on V^* satisfy the Yetter-Drinfeld condition we can define a braiding $\sigma : V^* \otimes V^* \rightarrow V^* \otimes V^*$ by the usual formula. The braiding on V and V^* are connected by the following formula, for $\alpha, \beta \in V^*$ and $v, w \in V$:

$$\text{ev}(\text{id} \otimes \text{ev} \otimes \text{id})(\alpha \otimes \beta \otimes \sigma(v \otimes w)) = \text{ev}(\text{id} \otimes \text{ev} \otimes \text{id})(\sigma(\alpha \otimes \beta) \otimes v \otimes w) . \quad (3)$$

Remark 9.2 By using equation (3) we can identify $(L^1)^{\wedge n}$ with the linear maps from $\mathfrak{p}^{\wedge n}$ to k . To avoid unnecessary crossings in the braided category we use the evaluation which pairs off the elements from the inside, i.e. $\text{ev}(\text{id} \otimes \text{ev} \otimes \text{id}) : \mathfrak{p} \wedge \mathfrak{p} \otimes L^1 \wedge L^1 \rightarrow k$ etc., which we just write $\underline{\text{ev}}$.

If \mathfrak{p} is finite dimensional, the map $\text{id} \otimes \text{id} - \sigma : \mathfrak{p} \wedge \mathfrak{p} \rightarrow \mathfrak{p} \wedge \mathfrak{p}$ is invertible, and we write $T = [\cdot] \circ (\text{id} \otimes \text{id} - \sigma)^{-1} : \mathfrak{p} \wedge \mathfrak{p} \rightarrow \mathfrak{p}$. For example if σ has eigenvalues ± 1 (the commutative case) then $T(\alpha \wedge \beta) = \frac{1}{2}[\alpha, \beta]$. In the general case, Proposition 8.3 gives

$$\underline{\text{ev}}((\alpha \wedge \beta) \otimes d\xi) = T(\alpha \wedge \beta)(\xi) . \quad (4)$$

We can continue this formula, e.g.

$$\begin{aligned} \underline{\text{ev}}((\alpha \wedge \beta \wedge \gamma) \otimes d(\xi \wedge \eta)) &= \underline{\text{ev}}((\alpha \wedge \beta \wedge \gamma) \otimes (d\xi \wedge \eta - \xi \wedge d\eta)) \\ &= \underline{\text{ev}}(\alpha \wedge T(\beta \wedge \gamma) - T(\alpha \wedge \beta) \wedge \gamma)(\xi \wedge \eta) . \end{aligned} \quad (5)$$

Now we are in position to introduce the cohomology of a Hopf-Lie algebra.

Definition 9.3 Define a cochain complex by K^n being the linear maps from $\mathfrak{p}^{\wedge n}$ to k . The differential $d : K^n \rightarrow K^{n+1}$ is given by

$$\begin{aligned} d\phi(\alpha_0 \wedge \dots \wedge \alpha_n) &= \phi(\alpha_0 \wedge \dots \wedge T(\alpha_{n-1} \wedge \alpha_n)) - \dots \\ &\quad - (-1)^n \phi(T(\alpha_0 \wedge \alpha_1) \wedge \dots \wedge \alpha_n) . \end{aligned} \quad (6)$$

The cohomology $H_{HL}(\mathfrak{p})$ of this complex is known as the *Hopf-Lie cohomology* of \mathfrak{p} .

Proposition 9.4 *The Hopf-Lie cohomology $H_{HL}(\mathfrak{p})$ of \mathfrak{p} is isomorphic to the cohomology $H^s(\text{}^{coP}(\Omega^*P), d)$ of the left invariant forms.*

Proof Combining the results of this section and the last section.

Acknowledgements

T. Brzeziński would like to thank the Engineering and Physical Sciences Research Council for an Advanced Fellowship.

References

- [1] S. Amitsur, *Simple algebras and cohomology groups of arbitrary fields*. Trans. Amer. Math. Soc. 90 (1959), 73–112.
- [2] M. Artin, *On Azumaya algebras and finite representations of rings*. J. Algebra 11 (1969), 532–563.
- [3] E.J. Beggs, *The de Rham complex on infinite dimensional manifolds*. Quart. J. Math. Oxford (2), 38 (1987), 131–154.
- [4] E.J. Beggs, *Braiding and exponentiating noncommutative vector fields*. Preprint Arxiv:math.QA/0306094.
- [5] R.J. Blattner, M. Cohen & S. Montgomery, *Crossed products and inner actions of Hopf algebras*. Trans. Am. Math. Soc. 298 (1986), 671–711.
- [6] A. Borel & N. Wallach, *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*. Annals of Math. Studies 94, Princeton University Press, Princeton (1980).
- [7] K.S. Brown, *Cohomology of Groups*. Springer-Verlag, Berlin (1982).
- [8] T. Brzeziński, *Remarks on bicovariant differential calculi and exterior Hopf algebras*. Lett. Math. Phys. 27 (1993), 287–300.
- [9] T. Brzeziński & S. Majid, *Quantum group gauge theory on quantum spaces*. Comm. Math. Phys. 157 (1993), 591–638 (Erratum: 167 (1995), 235).
- [10] T. Brzeziński & S. Majid, *Quantum differentials and the q -monopole revisited*. Acta Appl. Math. 54 (1998), 185–232.
- [11] C. Chevalley & S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*. Trans. Am. Math. Soc. 63 (1948), 85–124.
- [12] Y. Doi & M. Takeuchi, *Cleft comodule algebras for a bialgebra*. Comm. Algebra 14 (1986), 801–817.
- [13] X. Gomez & S. Majid, *Braided Lie algebras and bicovariant differential calculi over co-quasitriangular Hopf algebras*. J. Algebra 261 (2003), 334–388.

- [14] I. Heckenberger & A. Schüler, *De Rham cohomology and Hodge decomposition for quantum groups*. Proc. Lond. Math. Soc. 83 (2001), 743-768.
- [15] N. Jacobson, *Lie Algebras*. Interscience Publishers, New York (1962).
- [16] A. Klimyk & K. Schmüdgen, *Quantum Groups and Their Representations*. Springer, Berlin (1997)
- [17] J. Madore, *An Introduction to Noncommutative Differential Geometry and Its Physical Applications*. 2nd ed., London Mathematical Society Lecture Note Series, 257, Cambridge University Press, Cambridge (1999).
- [18] S. Majid, *Quantum and braided Lie algebras*. J. Geom. Phys. 13 (1994), 307-356.
- [19] J. McCleary, *A User's Guide to Spectral Sequences*. 2nd ed., Cambridge University Press, Cambridge (2001).
- [20] A. Pressley & G.B. Segal, *Loop Groups*. Oxford University Press, Oxford (1986).
- [21] M.E. Sweedler, *Hopf Algebras*. Benjamin, New York (1969).
- [22] S.L. Woronowicz, *Differential calculus on compact matrix pseudogroups (quantum groups)*. Comm. Math. Phys. 122 (1989), 125–170.